MATH 2050C Lecture 25 (Apr 20)

Final: May 11 (Thurs) 9:30 am - 11:30 am

Topics to be covered: (refer to Bartle (4th Ed.))

- § 2.1 - 2.5 (except binary / decimal representations)
- § 3.1 - 3.5
- § 4.1 - 4.2
- § 5.1 - 5.4 (except approximation by step functions / polynomials)

REVIEW SESSION

Chapter 2 The Real Numbers

$\mathbb{R}$

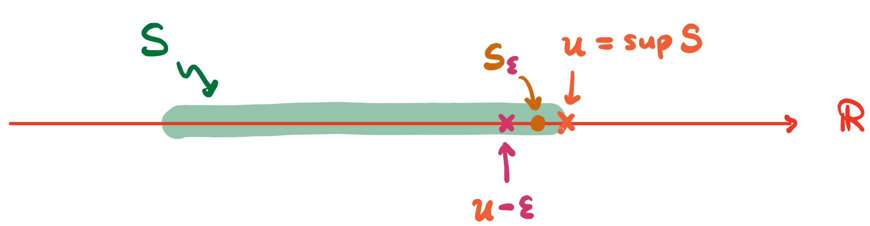
- **field** [§2.1]: $+, -, \times, \div$; field axioms
- **ordered** [§2.1-2.2]: "trichotomy" $a<b$, $a=b$, $a>b$
- **complete** [§2.3-2.4]: existence of $\sup S$, $\inf S$

Completeness Property [§2.3]: Every $\phi \neq S \subseteq \mathbb{R}$ that is bounded above has a supremum in $\mathbb{R}$.

Def [§2.3]:

Def$^{\underline{n}}$ [§2.3]: $u = \sup S \iff \begin{cases} u \geq s & \forall s \in S \\ \forall \varepsilon > 0, \ \exists s_\varepsilon \in S & \text{s.t. } u - \varepsilon < s_\varepsilon \end{cases}$

Picture:



! Caution ! $\sup S \in S$ OR $\sup S \notin S$

Useful Inequalities [§2.2]: AM-GM ineq., (reversed) triangle ineq., Bernoulli's ineq.

Useful Facts [§2.4] (from completeness):

- $\mathbb{N}$ is NOT bounded above (Archimedean Property)
- Density of $\mathbb{Q}$ and $\mathbb{R} \setminus \mathbb{Q}$ in $\mathbb{R}$
- Existence of $\sqrt{2}$

Intervals: $\boxed{\S 2.5}$

- characterization of intervals ("Connectedness")
- Nested Interval Property: ("compactness")

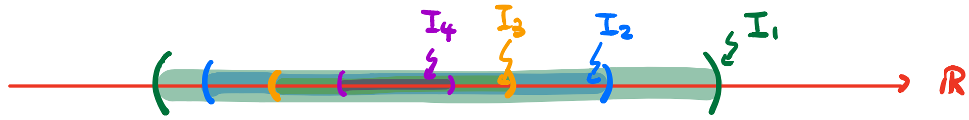
$$I_1 \supseteq I_2 \supseteq I_3 \supseteq \cdots\cdots \quad \Longrightarrow \quad \bigcap_{n=1}^{\infty} I_n \neq \phi$$

closed and bounded intervals

$\bigcap_{n=1}^{\infty} (0, \frac{1}{n}) = \phi$

$\bigcap_{n=1}^{\infty} [n, \infty) = \phi$

Picture:



Chapter 3 Sequences (and Series)

$\{x_n : n \in \mathbb{N}\}$ set $\neq$ seq. $(x_n) = (x_1, x_2, x_3, x_4, \ldots\ldots) : \mathbb{N} \to \mathbb{R}$

Def^{n}: $\boxed{\S 3.1}$

$\lim (x_n) = L \iff \forall \varepsilon > 0, \exists K \in \mathbb{N}$ s.t. (depends on ε)

$|x_n - L| < \varepsilon \quad \forall n \geq K$

$\boxed{\S 3.2}$

Limit Thm A: If $\lim(x_n)$ and $\lim(y_n)$ exist, then

- $\lim(x_n \pm y_n) = \lim(x_n) \pm \lim(y_n)$
- $\lim(x_n y_n) = \lim(x_n) \lim(y_n)$
- $\lim\left(\frac{x_n}{y_n}\right) = \frac{\lim(x_n)}{\lim(y_n)}$ ← Provided: $y_n \neq 0$, $\lim(y_n) \neq 0$

$(\frac{1}{n}) \to 0$

$\boxed{\S 3.2}$

Limit Thm B: If $\lim(x_n)$ and $\lim(y_n)$ exist, then

" $x_n \leq y_n \quad \forall n \in \mathbb{N} \quad \Longrightarrow \quad \lim(x_n) \leq \lim(y_n)$ "

[! Caution ! Only get "$\leq$" even if $x_n < y_n \ \forall n \in \mathbb{N}$. E.g. $0 < \frac{1}{n}$]

$\boxed{\S 3.2}$

FACT: (x_n) convergent $\Longrightarrow$ (x_n) bounded

+ monotone

Monotone Convergence Thm $\boxed{\S 3.3}$

$(x_n) = ((-1)^n)$

$(x_n) = (\frac{1}{n})$

$(x_n) = (\frac{(-1)^n}{n})$

To show (x_n) divergent	To show (x_n) convergent
(I) (x_n) unbounded §3.2	(I) ε-K definition §3.1
(II) $\exists$ two subseq of (x_n) §3.4	(II) Limit thms §3.2
$(x_{n_k}) \longrightarrow L$	(III) Squeeze thm §3.2
$\neq$	do NOT need to know the limit *(IV) Monotone Convergence Thm §3.3
$(x_{m_k}) \longrightarrow L'$	*(V) Cauchy criteria §3.5

Def$^{\underline{n}}$: §3.5

(x_n) is Cauchy $\iff$ $\forall \varepsilon > 0,\ \exists H \in \mathbb{N}$ s.t. (depends on ε)

$$|x_n - x_m| < \varepsilon \quad \forall\ \underline{n, m} \geq H$$

no relation between them

Cauchy Criteria: §3.5 (x_n) convergent $\iff$ (x_n) Cauchy

"iff"

Bolzano-Weierstrass Thm: §3.4 Any bounded seq has a convergent subseq.

[! Caution ! May have different subseq's converging to different limits.

E.g. $((-1)^n)$ $\rightsquigarrow$ limsup & liminf]

Chapter 4 Limits (of functions)

Setup: $f: A \longrightarrow \mathbb{R}$, $\underline{c \in \mathbb{R}}$ is a cluster pt of A

[! Caution ! Either $c \in A$ or $c \notin A$ is possible E.g.) $A = [0,1)$]

Def$^{\underline{n}}$: §4.1

$\lim_{x \to c} f(x) = L \iff \forall \varepsilon > 0,\ \exists \delta > 0$ s.t. (depends on ε)

$$|f(x) - L| < \varepsilon \quad \forall x \in \underline{A},\ \underbrace{0 < |x - c| < \delta}_{x \neq c}$$

Sequential Criteria: §4.1

$\lim_{x \to c} f(x) = L$ (limit of function) $\iff$ $\forall$ seq. (x_n) in $\underbrace{A \setminus \{c\}}_{x_n \neq c}$ s.t. $\lim(x_n) = c$, $\lim(f(x_n)) = L$ (limit of seq.)

[FACT: Useful to show $\lim_{x \to c} f(x)$ does NOT exist. E.g.) $f(x) = \sin\frac{1}{x}$]

- Limit Thm A and B carries over from seq. to functions $\boxed{\S 4.2}$

Chapter 5 Continuous Functions

$\boxed{\S 5.1}$

Def^{n}: $f : A \to \mathbb{R}$ is continuous at $c \in A$ $\iff$ $\forall \varepsilon > 0,\ \exists \delta > 0$ s.t. $|f(x) - f(c)| < \varepsilon \quad \forall x \in A,\ |x - c| < \delta$

depends on ε (and c)

no $0<$

[! Caution ! Unlike $\lim_{x \to c} f(x)$, we NEED $c \in A$ here.]

Sequential Criteria: $\boxed{\S 5.1}$

$f : A \to \mathbb{R}$ is cts at a cluster pt. $c \in A$ $\iff$ $\lim(f(x_n)) = f(c)$ $\forall$ seq. (x_n) in A s.t. $\lim(x_n) = c$

[FACT: Useful to show Discontinuity at c. E.g.) $f(x) = \begin{cases} 1 & , x \in \mathbb{Q} \\ 0 & , x \notin \mathbb{Q} \end{cases}$]

$\boxed{\S 5.2}$

Facts: f, g cts $\Rightarrow$ $f \pm g$, fg, f/g, $f \circ g$

composition

new!

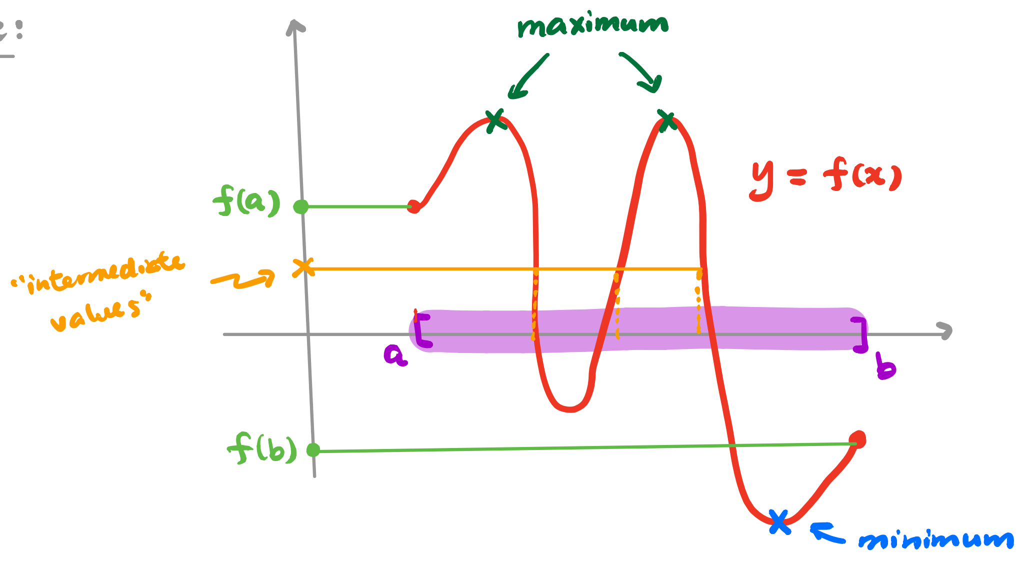
Two Theorems for cts $f : [a, b] \longrightarrow \mathbb{R}$ $\boxed{\S 5.3}$

closed + bdd

Extreme Value Thm: f achieves its absolute maximum and minimum.

Intermediate Value Thm: f achieves ALL intermediate values between $f(a)$ and $f(b)$.

Picture:



Def$^{\underline{n}}$: $\boxed{\S 5.4}$

$f : A \to \mathbb{R}$ is uniformly cts (on A) $\iff$ $\forall \varepsilon > 0,\ \exists \delta > 0$ s.t.

$|f(u) - f(v)| < \varepsilon \quad \forall u, v \in A,\ |u - v| < \delta$

depends ONLY on ε, but ~~NOT~~ u, v

FACT: f unif. cts on A $\Longrightarrow$ f cts on A (i.e. at ALL $c \in A$)

$\not\Longleftarrow$ $\because \delta$ may depend on $c \in A$

e.g.) $f(x) = x$ e.g. $f(x) = \frac{1}{x}$

Two Important Thm about uniform continuity $\boxed{\S 5.4}$

Uniform Continuity Thm:

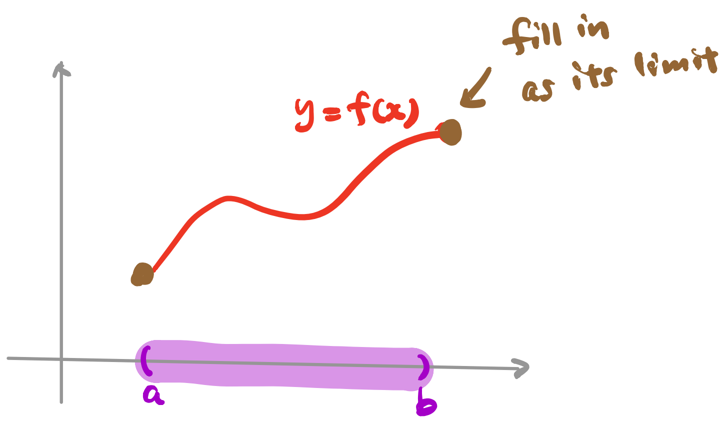
Any cts $f : [a, b] \to \mathbb{R}$ is uniformly cts.

closed + bdd

Continuous Extension Thm:

Any uniformly cts $f : (a, b) \to \mathbb{R}$ can be continuously extended to $[a, b]$.

Picture:



~ END OF REVIEW SESSION ~

Good Luck!